Growing wireless networks: preliminary back of the envelop calculations

Consider the following *Procedure 1* of growing a wireless network:

1: place a node, uniformly at random, on the unit square.
2: select a node uniformly at random from the already placed nodes.
3: pick a point, uniformly at random, from the disk centered at the selected node in 2 and place a new node there
4: go to 2

One can find the expected number of times the disk of a node will be selected as new nodes join the network. (From here on, selecting a node is equivalent to selecting a disk of a node.)

Let $E(s(v_1))$ be the expected number of times the first node in the network will be selected. With probability 1 the next node in the network, v_2, will select v_1; in turn, v_3 will select v_1 with probability $\frac{1}{2}$, v_4 will select v_1 with probability $\frac{1}{3}$ since there are now 3 nodes already in the network to choose from, etc.. Hence,

$E(s(v_1)) = 1 * 1 + 1 * \frac{1}{2} + 1 * \frac{1}{3} + \ldots + 1 * \frac{1}{n-1} = H_{n-1}$

Where H_n denotes the harmonic series of order n.

Similarly, one can determine $E(s(v_2))$: the third node in the network, v_3 will select v_2 with probability $\frac{1}{2}$ since it can select v_1 or v_2 equiprobably; v_4 will select v_2 with probability $\frac{1}{3}$ since there are now three nodes already in the network, etc.. Then,

$E(s(v_2)) = 1 * \frac{1}{2} + 1 * \frac{1}{3} + \ldots + 1 * \frac{1}{n-1} = H_{n-1} - H_{2-1} = H_{n-1} - H_1$

and for v_3 we have

$E(s(v_3)) = 1 * \frac{1}{3} + \ldots + 1 * \frac{1}{n-1} = H_{n-1} - H_2$

Then, by induction on j for $j > 1$,

$$E(s(v_j)) = H_{n-1} - H_{j-1}$$

Since j indexes time, the disks of nodes that have been present in the network longer would, in expectation, contain much larger number of nodes that joined the network later.

Arguing similarly one can find the expected number of nodes whose disks would not be selected. Note however that this is not relevant since they would still have neighbors (albeit potentially fewer) because of disks intersections.

First, consider the probability that the disk of node v_j will not be selected. v_1 will be selected by v_2 with probability 1. However, v_2 will *not* not be selected by the next node added to the network, v_3, with probability $\frac{1}{2}$ (v_3 can select equiprobably v_1 or v_2); v_4 will *not* select v_2 with probability $\frac{2}{3}$ (it can select v_3 or v_1 each with probability $\frac{1}{3}$). Similarly one can compute the probability that each of the remaining n-2 nodes added to the network will *not* select v_2. Since they will choose to select or *not* select v_2 independently of each other, the probability that none of them selects v_2 can be found using the probability product rule:

$$Pr(\forall\ j,\ 2 < j \leq n,\ no\ v_j\ selects\ v_2) = \frac{1}{2} * \frac{2}{3} * \frac{3}{4} * \ldots * \frac{n-2}{n-1} = \frac{1}{n-1}$$

Analogously, the probability that no node selects v_3 would be

$$Pr(\forall\ j,\ 3 < j \leq n,\ no\ v_j\ selects\ v_3) = \frac{2}{3} * \frac{3}{4} * \ldots * \frac{n-2}{n-1} = \frac{2}{n-1}$$

Or in general, the probability that no node selects v_k $(k \neq 1, k \neq n)$ is

$$Pr(\forall\ j,\ k < j \leq n,\ no\ v_j\ selects\ v_k) = \frac{k-1}{k} * \frac{k}{k+1} * \ldots * \frac{n-2}{n-1} = \frac{k-1}{n-1}$$

The probabilities that no node will select respectively v_{n-1} and v_n are $\frac{n-2}{n-1}$ and 1.

Now, the expected number of nodes, N, that will *not* be selected is given by

$$N = \frac{1}{n-1} + \frac{2}{n-1} + \ldots + \frac{n-2}{n-1} + \frac{n-1}{n-1} = \frac{1+2+3+\ldots+n-1}{n-1} = \frac{(n-1)n}{2(n-1)} = \frac{n}{2}$$

Is *Procedure 1* distinct from the following *Procedure 2*?

1: place a node, uniformly at random, on the unit square.
2: pick a point, uniformly at random, from the union of disks of nodes already in the network
3: go to 2

*Y*es, Procedure 1 is distinct from *Procedure 2* in general.

One can also consider the following *Procedure 3*:

1: place a node, uniformly at random, on the unit square.
2: toss a biased coin **3:** if Heads, select a node uniformly at random from the already placed nodes and pick a point, uniformly at random, from the disk centered at the selected node; if Tails, pick a random point in the unit square.
4: place a node at the selected point **5:** go to 2

The figures below demonstrate the resulting placements of nodes in a 20x20 square, for various network generative models.

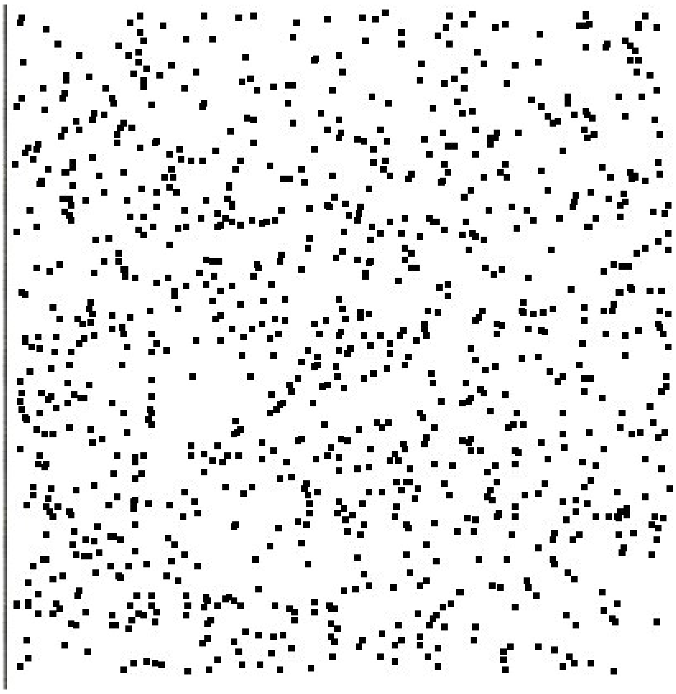

Figure 1. 2D Poisson point process. N=1000.

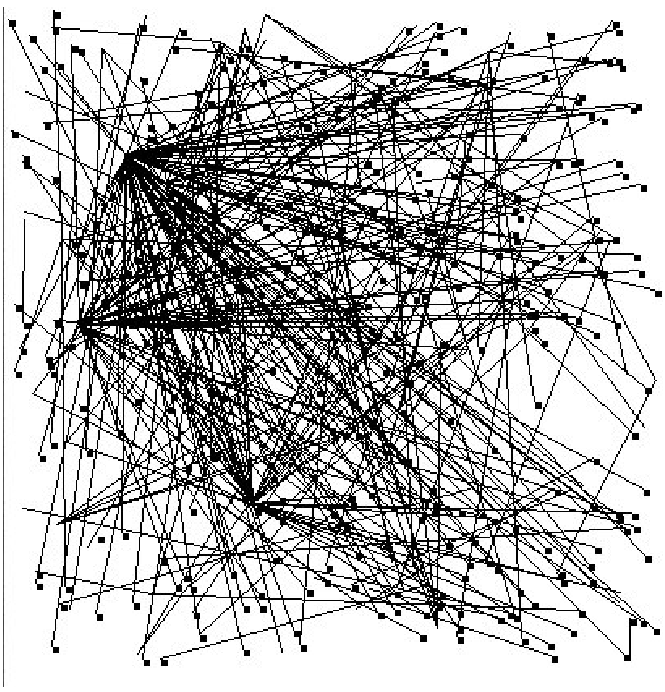

Figure 2. Preferential attachment. N=1000; p=0.6.

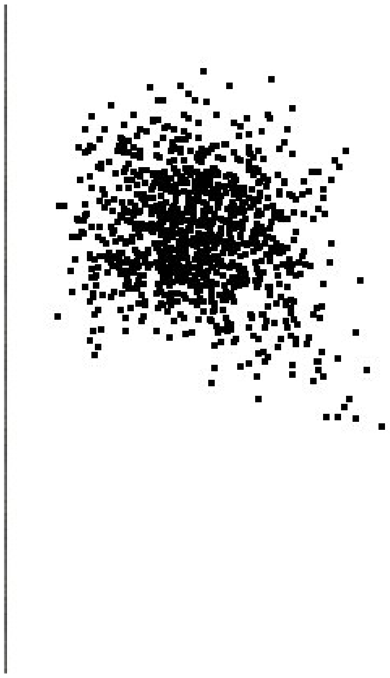

Figure 3. Procedure 1. N=1000; r=2.5.

Figure 4. Procedure 1. N=1000; r=5.

Figure 5. Procedure 3. N=1000; r=2.5; $P_{randomplacement}$=0.2.

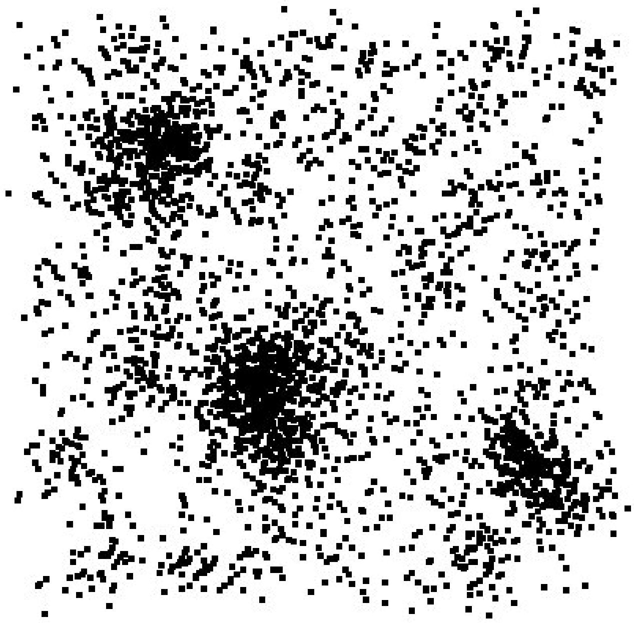

Figure 6. Procedure 3. N=3000; r=1; $P_{randomplacement}$=0.2.

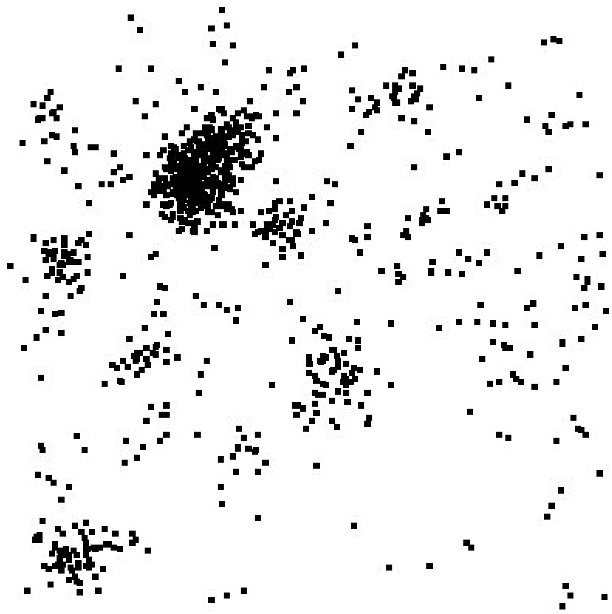

Figure 7. Procedure 3. N=1000; r=0.5; $P_{randomplacement}$=0.2.

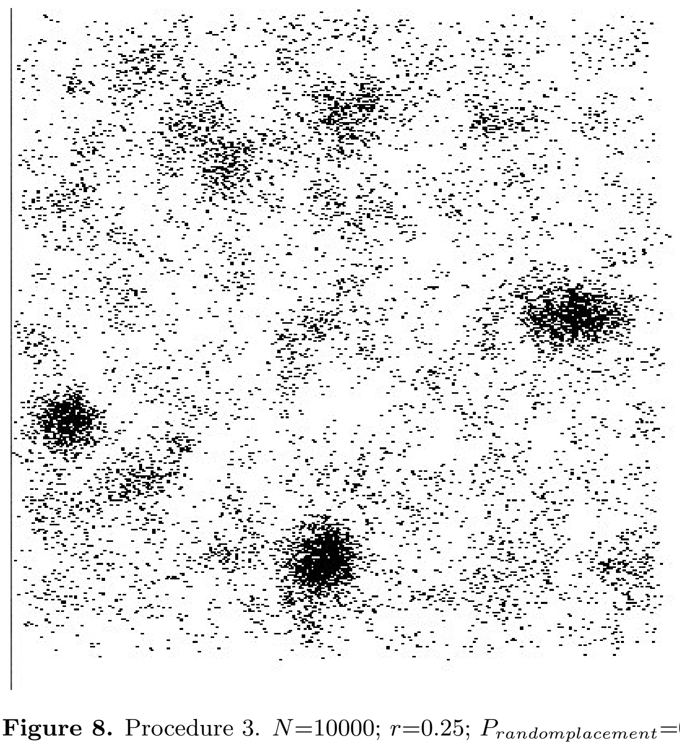

Figure 8. Procedure 3. N=10000; r=0.25; $P_{randomplacement}$=0.2.

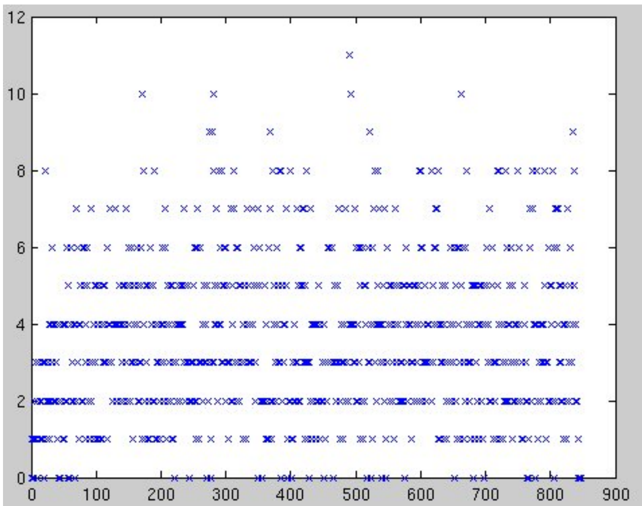

Figure 9. Procedure 1 degree frequency distribution; **x-axis:** degree; **y-axis:** frequency (number of nodes); N=3000; r=1.

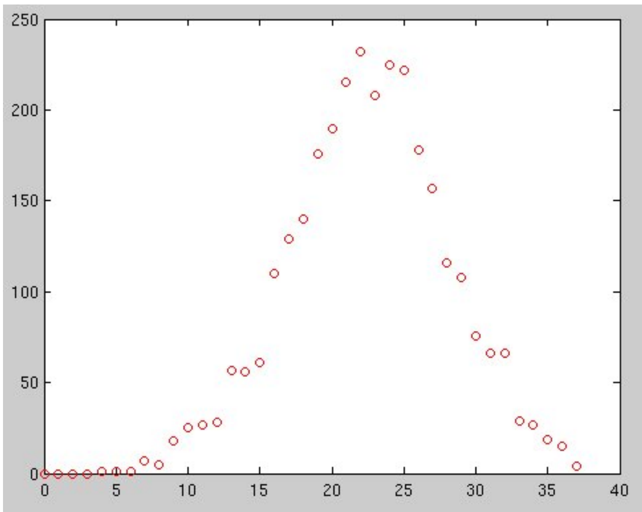

Figure 10. 2D Poisson point process degree frequency distribution; **x-axis:** degree; **y-axis:** frequency (number of nodes); N=3000; r=1.

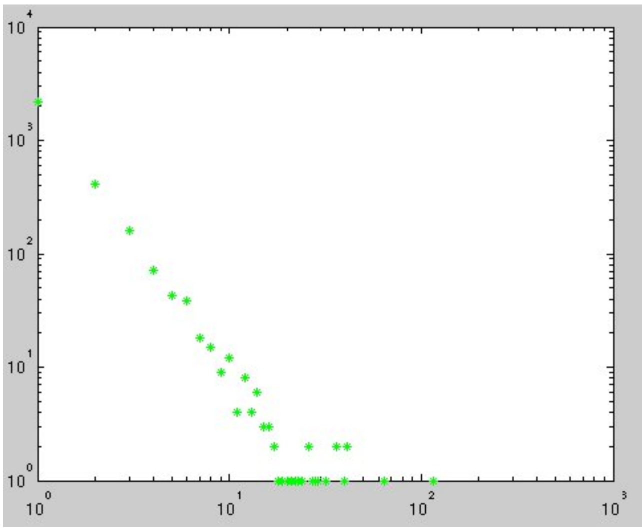

Figure 11. Preferential attachment process Log-Log degree frequency distribution; **x-axis:** degree; **y-axis:** frequency (number of nodes); N=3000; p=0.6.

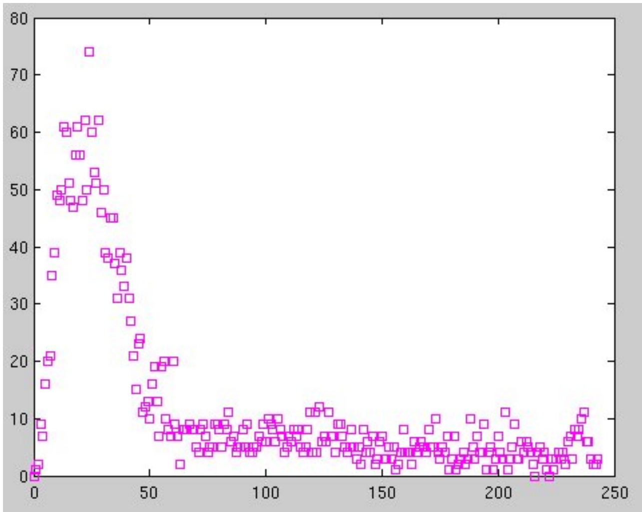

Figure 12. Procedure 3 frequency distribution; **x-axis:** degree; **y-axis:** frequency (number of nodes); N=3000; r=1; $P_{randomplacement}$=0.2.

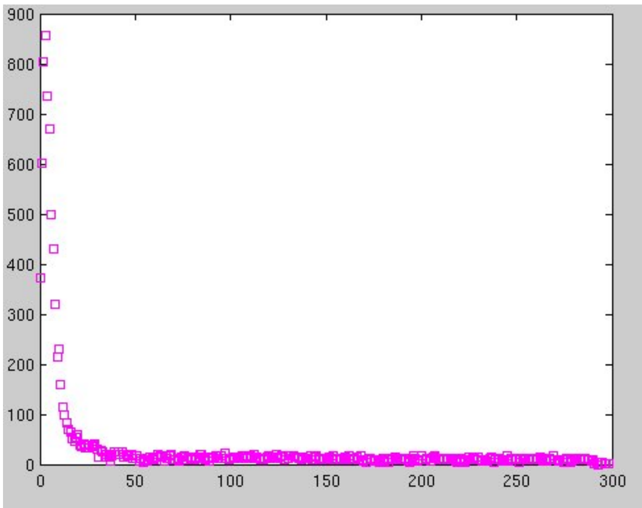

Figure 13. Procedure 3 ”zoomed out large scale” frequency distribution; **x-axis:** degree; **y-axis:** frequency (number of nodes); N=10000; r=0.25; $P_{randomplacement}$=0.2.

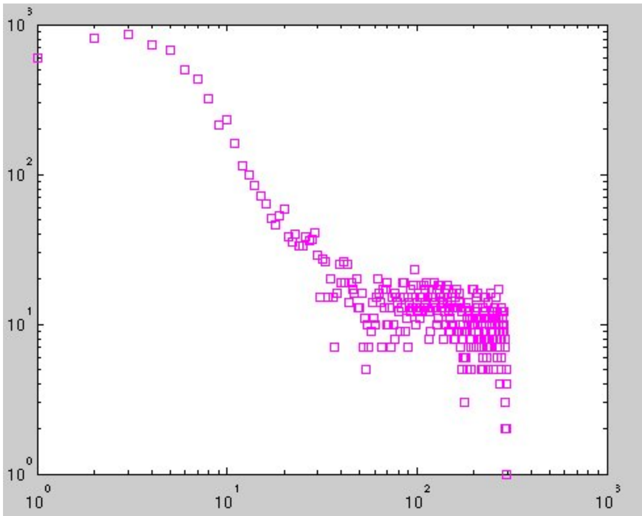

Figure 14. Procedure 3 ”zoomed out large scale” Log-Log frequency distribution; **x-axis:** degree; **y-axis:** frequency (number of nodes); N=10000; r=0.25; $P_{randomplacement}$=0.2.

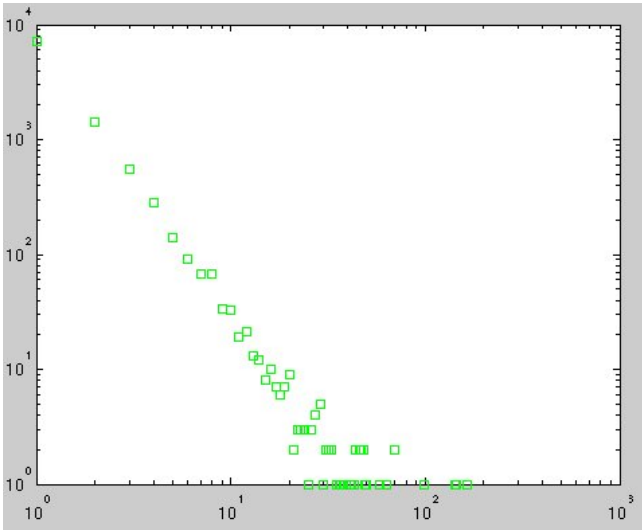

Figure 15. Preferential attachment Log-Log frequency distribution; **x-axis:** degree; **y-axis:** frequency (number of nodes); N=10000; p=0.6.

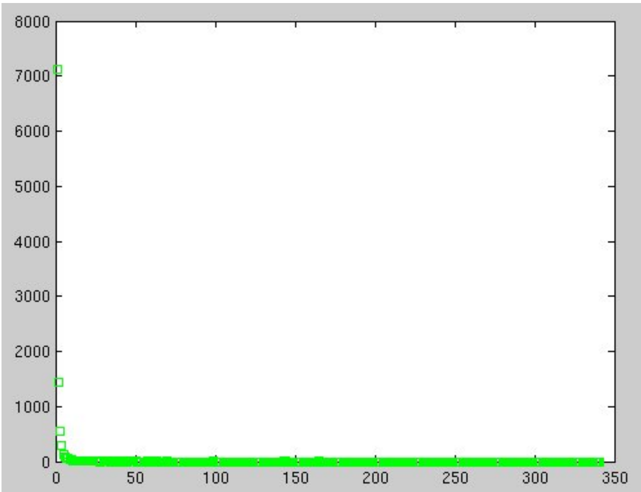

Figure 15. Preferential attachment frequency distribution; **x-axis:** degree; **y-axis:** frequency (number of nodes); N=10000; p=0.6.